

An Infinite-Dimensional Analogue of the Lebesgue Measure and Distinguished Properties of the Gamma Process

Natalia Tsilevich¹

*Department of Mathematics and Mechanics, St. Petersburg State University,
St. Petersburg, Russia*
E-mail: natalia@pdmi.ras.ru

Anatoly Vershik²

*Steklov Institute of Mathematics at St. Petersburg, 27 Fontanka, St. Petersburg,
191011 Russia*
E-mail: vershik@ndmi.ras.ru

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Marc Yor

*Laboratoire de Probabilités et Modèles aléatoires, Tour 56, 4 Place Jussieu,
75252 Paris Cedex 05, France*

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We define a one-parameter family \mathcal{L}_θ of sigma-finite (finite on compact sets) measures in the space of distributions. These measures are equivalent to the laws of the classical gamma processes and invariant under an infinite-dimensional abelian group of certain positive multipliers. This family of measures was first discovered by Gelfand–Graev–Vershik in the context of the representation theory of current groups; here we describe it in direct terms using some remarkable properties of the gamma processes. We show that the class of multiplicative measures coincides with the class of zero-stable measures which is introduced in the paper. We give also a new construction of the canonical representation of the current group $SL(2, \mathbb{R})^{\mathbb{Z}}$. © 2001 Academic Press

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1. INTRODUCTION

The main purpose of this work is to define and study a remarkable one-parameter family of sigma-finite measures \mathcal{L}_θ in an infinite-dimensional space of distributions. It was first discovered and used in the works by Gelfand–Graev–Vershik [14, 16, 5] on the representation theory of the group $SL(2, F)$, where F is an algebra of functions on a manifold. However in these works it was defined in a rather indirect and complicated way. We present here an explicit construction of these measures which allows to study them directly.

The starting point for our construction is the quasi-invariance of the gamma measure (the law of the gamma process) with respect to an infinite-dimensional group \mathcal{M} of multipliers by non-negative functions with summable logarithm. It generalizes in some way a well-known property of gamma distributions: if Y and Z are independent gamma variables, then the random variables $Y+Z$ and $Y/(Y+Z)$ are also independent (see Lemma 2.1). This remarkable quasi-invariance property has many applications, for example, it implies quasi-invariance of the Poisson–Dirichlet distributions with respect to some Markov operators, and gives an easy proof of the so-called Markov–Krein identity for distributions of means of random Dirichlet processes, see [13].

Given a quasi-invariant measure, the natural question is whether it admits an equivalent invariant measure. It happens that for the gamma processes the answer is positive. There exists a family of σ -finite (finite on compact sets) measures \mathcal{L}_θ which have an exponential density with respect to the laws of gamma processes and which are projective invariant with respect to the group \mathcal{M} and invariant under multiplications by functions with zero integral of logarithm. We call these measures *multiplicative*. We prove that each of the above invariance properties characterizes multiplicative measures in the class of σ -finite measures which are finite on compact sets and equivalent to Lévy processes.

From the general point of view the multiplicative measures generalize Lévy's theory to the case of σ -finite measures. There is a family $d\lambda_\theta = \frac{1}{\Gamma(\theta)} t^{\theta-1} dt$ of σ -finite measures on \mathbb{R}_+ , which we also call multiplicative, and which are related to the infinite-dimensional multiplicative measures \mathcal{L}_θ just the same way an ordinary infinitely divisible law is related to the Lévy process with the same Lévy measure. In particular, the measure \mathcal{L}_1 corresponds to the Lebesgue measure $\lambda = \lambda_1$, so we call \mathcal{L}_1 the *infinite-dimensional Lebesgue measure*. The invariance property of the measures \mathcal{L}_θ is a natural generalization of the invariance of the finite-dimensional multiplicative measures under the action of diagonal matrices with determinant one (a Cartan subgroup).

Multiplicative measures are closely related to stable processes and stable measures. We give an alternative definition of α -stable distributions which

is equivalent to the ordinary one but admits a reasonable limit as α goes to zero. It turns out that zero-stable measures in an appropriate sense are exactly the multiplicative measures.

In the spirit of [5] we can consider the multiplicative measure on the infinite-dimensional space (respectively, the multiplicative measure on the positive half-line) as the derivative of the α -stable law (respectively, the α -stable measure on the positive half-line) in parameter α at the point $\alpha = 0$. This corresponds to the derivative of the spherical function of the complementary series representation of the group $SL(2, \mathbb{R})$ with respect to the parameter at zero. But here we have a new interpretation of this effect: the multiplicative measure is in a sense a weak limit of stable laws; this property of the multiplicative measure was conjectured in [17]. The last link between stable laws and multiplicative measures can be illustrated by an almost trivial formula

$$\lim_{n \rightarrow \infty} \exp(-n(x^{1/n} - 1)) = \frac{d}{d\alpha} \exp(-x^\alpha)|_{\alpha=0} = \frac{1}{x},$$

which means that the Laplace transform of the zero-stable measure is the exponent of the derivative in α of the Laplace transform of α -stable laws at $\alpha = 0$.

An extremely important fact is the similarity between the end points of the interval $[0, 2]$: the point 2 corresponds to the gaussian measure which has (in an infinite-dimensional space) the spherical symmetry. The multiplicative measures have another infinite-dimensional linear group of symmetries—the group of multipliers. In between (for $\alpha \in (0, 2)$) the group of symmetries is unknown, but this group is not linear as for $\alpha = 0$ and $\alpha = 2$. Also it probably means that there is an analogue of the Itô–Malliavin calculus for the multiplicative measures.

The paper is organized as follows.

Section 2 contains a definition of the gamma process and its basic properties.

In Section 3 we prove the key property of the gamma process, namely its quasi-invariance with respect to a large group of multipliers.

Section 4 contains the key results of this paper. We introduce the family of infinite-dimensional multiplicative measures and study their main properties. We show that they are projective invariant with respect to the group \mathcal{M} of multipliers and invariant under the subgroup of multipliers by functions with zero integral of logarithm. We also prove that each of these properties characterizes multiplicative measures in the class of σ -finite measures equivalent to the laws of Lévy processes and finite on compact sets.

In Section 5 we use the results of the previous section to construct a model of the canonical representation of the current group $SL(2, F)$ in the $L^2(\mathcal{L}_1)$ space over a multiplicative measure. Namely, we describe the construction of [5] of the inductive limit of tensor products of complementary series representations in terms of multiplicative measures.

In Section 6 we give a definition of zero-stable measures and show that the class of zero-stable measures coincides with the class of multiplicative measures. We also prove that the gamma process is a weak limit of renormalized stable processes in an appropriate sense.

Appendix contains some basic properties of general Lévy processes. Theorem A.1 states that the law of a Lévy process is the product of its conic part (the measure on the cone of positive convergent series) and a product measure on sequences of points of the base space. Theorem A.2 is a characterization of measures on the cone which can be obtained as conic parts of Lévy processes. In some sense, these measures enjoy the greatest possible independence of coordinates.

The topics touched upon in this paper pose many new problems, only a small part of which is mentioned here.

2. THE GAMMA PROCESS

2.1. Lévy Processes on General Spaces

Let (X, ν) be a standard Borel space with a non-atomic finite non-negative measure ν , and let $\nu(X) = \tau$ be the total charge of ν . We denote by

$$D = \left\{ \sum z_i \delta_{x_i}, x_i \in X, z_i \in \mathbb{R}, \sum |z_i| < \infty \right\}$$

the real linear space of all finite real discrete measures on X , and by $D^+ = \{ \sum z_i \delta_{x_i} \in D : z_i > 0 \} \subset D$ the cone in D consisting of all positive measures.

Consider a class of measures \mathcal{A} on the half-line \mathbb{R}_+ satisfying the following conditions,

$$\mathcal{A}(0, \infty) = \infty, \quad (1)$$

$$\mathcal{A}(1, \infty) < \infty, \quad (2)$$

$$\int_0^1 s d\mathcal{A}(s) < \infty, \quad (3)$$

$$\mathcal{A}(\{0\}) = 0, \quad (4)$$

that is, Λ is the Lévy measure of a “non-trivial” (i.e. non-compound Poisson) subordinator.

Let ψ_Λ be the Laplace transform of the infinitely divisible distribution F_Λ with Lévy measure Λ :

$$\psi_\Lambda(t) = \exp \left(- \int_0^\infty (1 - e^{-ts}) d\Lambda(s) \right).$$

Each bounded Borel function $a: X \rightarrow \mathbb{R}$ defines a linear functional f_a on D , where $f_a(\eta) = \int_X a(x) d\eta(x)$ for $\eta \in D$.

DEFINITION 2.1. A Lévy process on the space (X, ν) with Lévy measure Λ satisfying (1)–(4) is a generalized process on D whose law P_Λ has Laplace transform

$$\mathbb{E} \left[\exp \left(- \int_X a(x) d\eta(x) \right) \right] = \exp \left(\int_X \log \psi_\Lambda(a(x)) d\nu(x) \right), \quad (5)$$

where a is an arbitrary non-negative bounded Borel function on X .

The correctness of this definition is guaranteed by the following explicit construction (see [8, Chapter 8]). Consider a Poisson point process on the space $X \times \mathbb{R}_+$ with mean measure $\nu \times \Lambda$. We associate with a realization $\Pi = \{(X_i, Z_i)\}$ of this process the element

$$\eta = \sum_{(X_i, Z_i) \in \Pi} Z_i \delta_{X_i} \in D. \quad (6)$$

Then η is a random discrete measure obeying the law P_Λ . In particular, it follows that the law P_Λ of the Lévy process is concentrated on the cone D^+ .

Remark 2.1.

1. Our definition of Lévy processes is closely related to the notion of a completely random measure, see [6], [8, Chapter 8].

2. If X is an interval in \mathbb{R}_+ and ν is the Lebesgue measure, we recover the ordinary definition of a *subordinator* (a process with stationary positive independent increments) corresponding to the Lévy measure Λ .

3. It is easy to see that for any constant $c > 0$ we have $P_\Lambda(\nu) = P_{c\Lambda}(\nu/c)$.

4. Let $L: X \rightarrow X$ be a ν -preserving transformation of the space X . It acts on D^+ by substitutions: $L(\sum z_i \delta_{x_i}) = \sum z_i \delta_{Lx_i}$. Obviously, the distribution P_Λ of a Lévy process is invariant under these transformations. This simple consideration plays an important role in the sequel.

See Appendix for some basic properties of Lévy processes, in particular, the Decomposition Theorem and definition of the conic part.

2.2. The Gamma Process

DEFINITION 2.2. *The gamma process with shape parameter $\theta > 0$ and scale parameter $\beta > 0$ on the space (X, ν) is a Lévy process on (X, ν) with Lévy measure $d\Lambda_{\theta, \beta}(z) = \theta z^{-1} e^{-\beta z} dz$, $z > 0$.*

The corresponding infinitely divisible law is the gamma distribution on \mathbb{R}_+ with shape parameter θ and scale parameter β , i.e.

$$\frac{\beta^\theta}{\Gamma(\theta)} t^{\theta-1} e^{-\beta t} dt, \quad t > 0.$$

Note that if η^β is the gamma process with scale parameter β , then $\eta^\beta = \beta\eta$, where η is the gamma process with scale parameter 1 (equality in distribution). In view of Theorem 3.1 below the law of η^β is equivalent to the law of η . Thus we will consider only gamma processes with scale parameter 1.

The law \mathcal{G}_θ of the gamma process (called the *gamma measure* with parameter θ on the space (X, ν)) is thus given by the Laplace transform

$$\mathbb{E}_{\mathcal{G}_\theta} \left[\exp \left(- \int_X a(x) d\eta(x) \right) \right] = \exp \left(- \theta \int_X \log(1 + a(x)) d\nu(x) \right), \quad (7)$$

where a is an arbitrary non-negative bounded Borel function on X .

Note that in view of Remark 3 above the conic part of the gamma process (see Appendix) depends only on the product of the shape parameter θ of the gamma distribution and the total charge $|\nu|$ of the parameter measure ν .

It follows from the Poisson construction (6) and Campbell's theorem on sums over Poisson processes (see [8, 3.2]) that each function $a: X \rightarrow \mathbb{R}$ with $\int_X \log(a(x) + 1) d\nu(x) < \infty$ correctly defines a measurable linear functional $\eta \mapsto f_a(\eta) = \int_X a(x) d\eta(x)$ on D with respect to \mathcal{G}_θ , and formula (7) holds for all such functions a .

It is well known that the gamma distribution enjoys the following property. If Y and Z are independent gamma variables with the same scale parameter, then the variables $Y + Z$ and $\frac{Y}{Y+Z}$ are independent. Moreover, a remarkable result of Lukacs [10] (similar to the famous Bernstein's characterization of normal distributions) states that this property is characteristic of the gamma distribution, i.e. if Y and Z are independent non-degenerate positive random variables, and the variables $Y + Z$ and $\frac{Y}{Y+Z}$ are independent, then both Y and Z have gamma distributions with the same scale parameter. These results imply the corresponding statements for

the gamma process which are key points for many important properties of \mathcal{G}_θ .

LEMMA 2.1. (1) *The total charge $\gamma(X)$ of the gamma process and the normalized gamma process $\bar{\gamma} = \gamma/\gamma(X)$ are independent. The distribution of the total charge is the gamma distribution with shape parameter $|\nu| \theta$, where $|\nu| = \nu(X)$ is the total charge of the measure ν .*

(2) *If for some Lévy process the total charge and the normalized process are independent, then it is a gamma process (maybe with some scale parameter).*

3. QUASI-INVARIANCE PROPERTY OF THE GAMMA PROCESS

Let $\mathcal{M} = \mathcal{M}(X, \nu)$ be the set of (classes mod 0 of) non-negative measurable functions on the space X with ν -summable logarithm,

$$\mathcal{M} = \left\{ a: X \rightarrow \mathbb{R}_+ : \int_X |\log a(x)| d\nu(x) < \infty \right\}.$$

Each function $a \in \mathcal{M}$ defines not only a linear functional f_a on D but also a multiplier $M_a: D \rightarrow D$, where $(M_a \eta)(x) = a(x) \eta(x)$, that is $M_a \eta = \sum a(x_i) z_i \delta_{x_i}$ for $\eta = \sum z_i \delta_{x_i}$. Note that the set \mathcal{M} is a commutative group with respect to pointwise multiplication of functions, and M_a is a group action of \mathcal{M} .

The following property of the gamma process was first discovered in [16, 5] in quite different terms; it plays an important role in the representation theory of the current group $SL(2, F)$, where F is the space of functions on a manifold.

THEOREM 3.1. *For each $a \in \mathcal{M}$, the gamma measure \mathcal{G}_θ with parameter θ is quasi-invariant under M_a , and the corresponding density is given by the following formula,*

$$\frac{d(M_a \mathcal{G}_\theta)}{d\mathcal{G}_\theta}(\eta) = \exp \left(-\theta \int_X \log a(x) d\nu(x) \right) \cdot \exp \left(- \int_X \left(\frac{1}{a(x)} - 1 \right) d\eta(x) \right).$$

Proof. Fix $a \in \mathcal{M}$ and let $\xi = M_a \eta$. Consider an arbitrary function $b \in \mathcal{M}$. Then $f_b(\xi) = \int_X b(x) d\xi(x) = \int_X a(x) b(x) d\eta(x) = f_{ab}(\eta)$. Thus, in view of (7), the Laplace transform $\mathbb{E}[\exp(-f_b(\xi))]$ with respect to \mathcal{G}_θ equals

$$\begin{aligned}
\mathbb{E}[\exp(-f_{ab}(\eta))] &= \exp\left(-\theta \int_X \log\left(1 + a(x)b(x)\right) dv(x)\right) \\
&= \exp\left(-\theta \int_X \log a(x) dv(x)\right) \\
&\quad \cdot \exp\left(-\theta \int_X \log\left(\frac{1}{a(x)} + b(x)\right) dv(x)\right).
\end{aligned}$$

Using (7) once more, we may consider the last factor as the Laplace transform of the gamma measure \mathcal{G}_θ calculated on the function $(\frac{1}{a(x)} - 1) + b(x)$. Let $I(a) = \exp(-\theta \int_X \log a(x) dv(x))$. Then we have

$$\begin{aligned}
\mathbb{E}[\exp(-f_b(\xi))] &= I(a) \cdot \mathbb{E}\left[\exp\left(-\int_X \left(\frac{1}{a(x)} - 1 + b(x)\right) d\eta(x)\right)\right] \\
&= \mathbb{E}\left[I(a) \cdot \exp\left(-\int_X \left(\frac{1}{a(x)} - 1\right) d\eta(x)\right)\right. \\
&\quad \left.\cdot \exp\left(-\int_X b(x) d\eta(x)\right)\right],
\end{aligned}$$

and Theorem 3.1 follows. ■

In particular, if we consider multiplication by a constant $c > 0$, then the corresponding density depends only on the total charge $\eta(X)$ of η , namely

$$\frac{d(M_c \mathcal{G}_\theta)}{d\mathcal{G}_\theta}(\eta) = \frac{1}{c^{\theta|\eta|}} \cdot \exp\left(\left(1 - \frac{1}{c}\right)\eta(X)\right). \quad (8)$$

Note that in fact this result follows from the independence property of the gamma process (Lemma 2.1).

THEOREM 3.2. *The action of the group \mathcal{M} on the space $(D^+, \mathcal{G}_\theta)$ is ergodic.*

Proof. Let $G: D^+ \rightarrow \mathbb{R}$ be a \mathcal{G}_θ -measurable functional on D^+ which is invariant under all M_a , i.e. $G(M_a \eta) = G(\eta)$ a.e. with respect to \mathcal{G}_θ . Consider an arbitrary Borel function $k: \mathbb{R} \rightarrow \mathbb{R}$. Then for each $a \in \mathcal{M}$

$$\begin{aligned}
\mathbb{E}_\theta[k(G(\eta))] &= \mathbb{E}_\theta[k(G(M_a \eta))] \\
&= \mathbb{E}_\theta\left[k(G(\eta)) \cdot \exp\left(-\int_X \tilde{a}(x) d\eta(x)\right)\right] \\
&\quad \cdot \exp\left(-\theta \int_X \log a(x) dv(x)\right),
\end{aligned}$$

where $\tilde{a}(x) = (1/a(x)) - 1$, and \mathbb{E}_θ denotes the expectation with respect to \mathscr{G}_θ . But in view of (7) the last factor equals

$$\left(\mathbb{E}_\theta\left[\exp\left(-\int_X \tilde{a}(x) \, d\eta(x)\right)\right]\right)^{-1} = (\mathbb{E}_\theta[\exp(-f_{\tilde{a}}(\eta))])^{-1};$$

hence we have

$$\mathbb{E}_\theta[k(G(\eta)) \exp(-f_{\tilde{a}}(\eta))] = \mathbb{E}_\theta[k(G(\eta))] \cdot \mathbb{E}_\theta[\exp(-f_{\tilde{a}}(\eta))].$$

Thus G is independent of every functional f_a , and Theorem 3.2 follows. ■

A natural question is whether the quasi-invariance property stated in Theorem 3.1 is characteristic of the gamma process. The answer to this question is negative. An example of a quasi-invariant Lévy process which is not equivalent to any gamma process is constructed in the work [11] which appeared after the preliminary version of this paper had been published. The situation is similar to the quasi-invariance property with respect to translations on appropriate vectors—not only gaussian measure has this property. But the next question is whether there exist quasi-invariant measures which admit equivalent invariant measures. And unlike the gaussian case, it turns out that the answer is positive only for gamma processes—only processes equivalent to gamma processes admit an equivalent invariant σ -finite measure (see Corollaries 4.2 and 4.3 below).

4. MULTIPLICATIVE MEASURES AND THE INFINITE-DIMENSIONAL LEBESGUE MEASURE

4.1. Infinite-Dimensional Multiplicative Measures

The question whether the law of the gamma process admits an equivalent invariant measure leads to the following definition of a one-parameter family of σ -finite measures on the space D^+ . They were first discovered in [16, 5] in quite a different context.

DEFINITION 4.1. The *multiplicative measure* with parameter $\theta > 0$ on the space $D^+(X, \nu)$ is a σ -finite measure (finite on compact sets) \mathscr{L}_θ which is equivalent to the law of the gamma process with parameter θ and is defined by the following formula:

$$\frac{d\mathscr{L}_\theta}{d\mathscr{G}_\theta}(\eta) = \exp(\eta(X)). \tag{9}$$

The multiplicative measure with parameter 1 is called *the Lebesgue measure* on $D^+(X, \nu)$.

Remark 4.1. Since we deal now with σ -finite measures, they are defined up to a constant $c > 0$ which could be fixed, for example, by normalizing on some compact set. All statements concerning σ -finite measures should be understood taking this remark into account.

It follows from formulae (7) and (9) that the Laplace transform of the multiplicative measure \mathcal{L}_θ equals

$$\int_{D^+} \left[\exp \left(- \int_X a(x) d\eta(x) \right) \right] d\mathcal{L}_\theta(\eta) = \exp \left(- \theta \int_X \log a(x) d\nu(x) \right). \quad (10)$$

Note that this transform is defined only on the class \mathcal{M} of functions with summable logarithm, which is not a linear space.

The transition from the gamma measure \mathcal{G}_θ to the corresponding multiplicative measure \mathcal{L}_θ does not change the conditional measures given the full charge $\eta(X)$ equal to some $s > 0$. We change only the factor measure on \mathbb{R}_+ that is the distribution of $\eta(X)$. For \mathcal{G}_θ this factor measure is the gamma distribution with density $\frac{1}{\Gamma(\theta)} t^{\theta-1} e^{-t}$, and for \mathcal{L}_θ it is a measure λ_θ with the gamma density multiplied by e^t , that is

$$\frac{d\lambda_\theta(t)}{dt} = \frac{1}{\Gamma(\theta)} t^{\theta-1}. \quad (11)$$

We call λ_θ the one-dimensional *multiplicative measure* with parameter $\theta > 0$. In particular case $\theta = 1$ the multiplicative measure is just the Lebesgue measure. Note that these measures are σ -finite but finite on compact sets.

The Laplace transform of the multiplicative measure λ_θ on \mathbb{R}_+ equals $\phi_\theta(t) = 1/t^\theta$. Thus it follows from (10) that

$$\int_{D^+} \exp(-f_a(\eta)) d\mathcal{L}_\theta(\eta) = \exp \left(\int_X \log \phi_\theta(a(x)) d\nu(x) \right),$$

and a comparison with (5) shows that *the infinite-dimensional multiplicative measure \mathcal{L}_θ is related to the one-dimensional multiplicative measure λ_θ just in the same way as the law of a Lévy process is related to the infinite divisible law with the same Lévy measure*. Thus our work contains some hint to a theory of σ -finite infinitely divisible measures.

Note that the multiplicative measures \mathcal{L}_θ , as well as the corresponding one-dimensional measures λ_θ , form a semigroup with respect to convolution, $\lambda_{\theta_1} * \lambda_{\theta_2} = \lambda_{\theta_1 + \theta_2}$ and $\mathcal{L}_{\theta_1} * \mathcal{L}_{\theta_2} = \mathcal{L}_{\theta_1 + \theta_2}$.

The following Theorem 4.1 states the key property of the multiplicative measures, that is their invariance under a very large group of transformations. But first we give a definition which is suggested by the representation theory and was used first in [5].

DEFINITION 4.2. Let (Y, μ) be a measurable space, and let G be a group acting on this space by transformations T_g , $g \in G$. Then the measure μ is called *projective invariant* with respect to G if it is quasi-invariant under all transformations T_g , and the densities

$$\frac{dT_g \mu}{d\mu}(y) = c(g)$$

are constant functions.

THEOREM 4.1. *The multiplicative measure \mathcal{L}_θ is projective invariant with respect to the group of multipliers \mathcal{M} , namely*

$$\frac{dM_a(\mathcal{L}_\theta)}{d\mathcal{L}_\theta} = \exp\left(-\theta \int_X \log a(x) d\nu(x)\right). \quad (12)$$

Proof. Follows from Theorem 3.1. ■

Let $\mathcal{M}_0 = \{a \in \mathcal{M} : \int_X \log a(x) d\nu(x) = 0\}$ be the subgroup of \mathcal{M} consisting of functions with zero integral of logarithm.

COROLLARY 4.1. *Multiplicative measures are invariant with respect to the subgroup \mathcal{M}_0 .*

Note that Theorem 4.1 and Corollary 4.1 are analogues of the corresponding properties of finite-dimensional multiplicative measures. Namely, consider the product $\lambda_\theta^n = \lambda_\theta \times \cdots \times \lambda_\theta$ (n factors), where $d\lambda_\theta = \frac{1}{t(\theta)} t^{\theta-1} dt$ is a multiplicative measure. It is easy to see that the measure λ_θ^n is *projective invariant* under the action of the diagonal subgroup of $\mathrm{GL}(n, \mathbb{R})$ and *invariant* under the action of the diagonal subgroup of $\mathrm{SL}(n, \mathbb{R})$. Moreover, the following easy lemma holds.

LEMMA 4.1. (1) *Let m be a measure on \mathbb{R}^n which is projective invariant with respect to the diagonal subgroup of $\mathrm{GL}(n, \mathbb{R})$. Then m is a multiplicative measure.*

(2) *Let m be a product measure on \mathbb{R}^n which is invariant with respect to the diagonal subgroup of $\mathrm{SL}(n, \mathbb{R})$. Then m is a multiplicative measure.*

4.2. Uniqueness Theorems

It turns out that infinite-dimensional multiplicative measures also enjoy the corresponding uniqueness properties.

Let us call a σ -finite measure μ on $D^+(X, \nu)$ *admissible*, if (1) it is equivalent to the law of some Lévy process; and (2) it is finite on compact sets.

Note that multiplicative measures satisfy these conditions.

THEOREM 4.2. *Let μ be an admissible measure on D^+ which is projective invariant with respect to the group \mathcal{M} . Then μ is a multiplicative measure.*

COROLLARY 4.2. *If a Lévy process admits an equivalent projective invariant measure, then it is equivalent to some gamma process.*

Proof of Theorem 4.2 Let π be an arbitrary measurable partition $X = X_1 \cup X_2$ of the space X . For $t_1, t_2 > 0$, let $L(t_1, t_2)$ be the value of the Laplace transform of the measure μ calculated on the function $t(x) = t_1 \chi_{X_1}(x) + t_2 \chi_{X_2}(x)$, where χ_A denotes the indicator of the set A . First, note that in view of the projective invariance of μ this value is finite. Indeed, consider the sets $A_k = \{\eta \in D^+ : 2^k \leq \eta(X) < 2^{k+1}\}$, and let $c = (dM_2\mu)/d\mu$, where M_2 is the multiplier by a constant function $a \equiv 2$. Since μ is finite on compact sets, $c \geq 1$. And since $M_2 A_k = A_{k+1}$, we have $\mu A_{k+1} = c\mu A_k$, hence $\mu A_k = c^k \cdot h$, where $h = \mu A_1$. Then

$$\int_{D^+} \exp(-t\eta(X)) d\mu(\eta) \leq \sum_{k \in \mathbb{Z}} hc^k \exp(-t2^k) < \infty.$$

Thus the Laplace transform of μ is finite on constant functions and it follows easily that it is finite on step functions, as desired.

It follows from the projective invariance of μ that

$$L(a_1 t_1, a_2 t_2) = d_1(a_1) d_2(a_2) L(t_1, t_2), \quad (13)$$

where $d_k(a_k) = (dM_{a_k}\mu)/d\mu$. Now a standard argument shows that the solutions of this functional equation are of the form $L(t_1, t_2) = \text{const} \cdot t_1^{-\theta_1} t_2^{-\theta_2}$ for some constants $\theta_1, \theta_2 \in \mathbb{R}$. It means that $L(t_1, t_2)$ is the Laplace transform of a product measure $\mu_1 \times \mu_2$ with multiplicative factors:

$$\frac{d\mu_k(t)}{dt} = \frac{c_k}{\Gamma(\theta_k)} t^{\theta_k-1},$$

and since μ is finite on compact sets, we have $\theta_k > 0$. It is clear that the measure μ_k depends only on the value $\tau = \nu(X_k)$. Denote it by μ_τ and its

parameters by $c(\tau)$ and $\theta(\tau)$, so that its Laplace transform equals $L_\tau(s) = c(\tau) s^{-\theta(\tau)}$. Since we have shown that the two-dimensional measure associated with every partition is a product measure, and $\eta(X_1 \cup X_2) = \eta(X_1) + \eta(X_2)$, we must have $L_{\tau_1 + \tau_2}(s) = L_{\tau_1}(s) L_{\tau_2}(s)$, that implies $c(\tau_1 + \tau_2) = c(\tau_1) c(\tau_2)$ and $\theta(\tau_1 + \tau_2) = \theta(\tau_1) + \theta(\tau_2)$. Thus $c(\tau) = c^\tau$ and $\theta(\tau) = \theta\tau$ for some constants $c, \theta > 0$.

Considering the Laplace transform of the measure μ first on step functions and then extending the resulting formula to arbitrary Borel functions in a standard way, we obtain that it equals

$$c \cdot \exp \left(-\theta \int_X \log a(x) dv(x) \right),$$

i.e. μ is a multiplicative measure with parameter θ . ■

Note that the first part of the proof is just one possible way to define finite-dimensional projections of the measure μ . For probability measures on D^+ an n -dimensional projection is the distribution of the random vector $(\eta(X_1), \dots, \eta(X_n))$, where $X = X_1 \cup \dots \cup X_n$ is some measurable partition π of the base space X . But this definition fails for non-finite measures. However, we still may define projections as follows. Define a function $L: \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ as the Laplace transform of μ calculated on the n -dimensional space of step functions generated by the indicator functions of the sets X_1, \dots, X_n (see the case $n = 2$ in the proof). Assume that this function is a.e. finite, and it is a Laplace transform of some measure μ_π in \mathbb{R}^n . Then μ_π is called the n -dimensional projection of the measure μ associated with the partition π . It is easy to see that for probability measures this definition is equivalent to the ordinary one.

Thus we obtain that the *finite-dimensional projection of the multiplicative measure \mathcal{L}_θ* associated with the partition $X = X_1 \cup \dots \cup X_n$, where $v(X_k) = \tau_k$, is the *product of one-dimensional multiplicative measures*:

$$\frac{d\mathcal{G}_\theta^\pi(t_1, \dots, t_n)}{dt_1 \cdots dt_n} = \prod_{k=1}^n \frac{1}{\Gamma(\tau_k \theta)} t_k^{\tau_k \theta - 1}. \quad (14)$$

This formula was first obtained in [5].

THEOREM 4.3. *Let μ be an admissible measure on D^+ which is invariant with respect to the subgroup \mathcal{M}_0 . Then μ is a multiplicative measure.*

COROLLARY 4.3. *If a Lévy process admits an equivalent \mathcal{M}_0 -invariant measure, then it is equivalent to some gamma process.*

Proof of Theorem 4.3. Since the measure μ is admissible, it can be written in the form $d\mu(\eta) = f(\eta) dP_A(\eta)$, where A is some Lévy measure.

Let π be an arbitrary measurable partition $X = X_1 \cup X_2$ of the space X , and denote $\nu_k = \nu|_{X_k}$, $k = 1, 2$. Then the space $D^+(X)$ decomposes in a natural way into the direct sum of $D^+(X_1)$ and $D^+(X_2)$, and the law P_A of the Lévy process is a product measure in this decomposition, that is $P_A^\nu(\eta) = P_A^{\nu_1}(\eta_1) \times P_A^{\nu_2}(\eta_2)$, where $\eta_k = \eta|_{X_k}$. One can easily see that the measure $f(\eta_k) dP_A^{\nu_k}(\eta_k)$ is invariant with respect to the group $\mathcal{M}_0(X_k)$, $k = 1, 2$. It follows that the measure $\tilde{\mu} = f(\eta_1) dP_A^{\nu_1}(\eta_1) \times f(\eta_2) dP_A^{\nu_2}(\eta_2) = f(\eta_1) f(\eta_2) dP_A(\eta)$ is invariant under the subgroup $\mathcal{M}_0(X_1) \times \mathcal{M}_0(X_2) \subset \mathcal{M}_0(X)$. Moreover, for any $a \in \mathcal{M}_0(X)$ there exists a ν -preserving transformation of X which sends a to an element of $\mathcal{M}_0(X_1) \times \mathcal{M}_0(X_2)$, thus in fact $\tilde{\mu}$ is also invariant under $\mathcal{M}_0(X)$.

Now consider the function $\tilde{L}(t_1, t_2)$ associated with the partition π as in the proof of Theorem 4.2, but for the measure $\tilde{\mu}$. Since it is a product measure in the decomposition $D^+(X) = D^+(X_1) \oplus D^+(X_2)$, we have $L(t_1, t_2) = L_1(t_1) L_2(t_2)$, and the invariance of $\tilde{\mu}$ under \mathcal{M}_0 implies that it is invariant under transformations of the form $(t_1, t_2) \mapsto (a_1 t_1, a_2 t_2)$ with $a_1 a_2 = 1$, i.e. $L_1(a t_1) L_2(\frac{1}{a} t_2) = L_1(t_1) L_2(t_2)$ for all $a > 0$. Then again a standard argument shows that $L(t_1, t_2) = \text{const} \cdot t_1^{-\theta_1} t_2^{-\theta_2}$ (and again this is a variant of Lemma 4.1, part 2), and the proof proceeds as in Theorem 4.2. ■

4.3. Multiplicative Measures on $D(X, \nu)$

We have defined the multiplicative measures on the cone $D^+(X)$ of positive discrete measures. Now we may extend this definition in a standard way to the whole space $D(X)$. Let $\mathcal{L}_\theta^+ = \mathcal{L}_\theta$. Consider the measure \mathcal{L}_θ^- on the space $D^- = \{\eta = \sum z_i \delta_{x_i} \in D : z_i < 0\}$ which is the image of \mathcal{L}_θ^+ under the mapping $\eta \rightarrow -\eta$.

DEFINITION 4.3. The *multiplicative measure* with parameter θ on the space $D(X)$ is the convolution $\mathcal{L}_\theta^+ * \mathcal{L}_\theta^-$. The *Lebesgue measure* on $D(X)$ is the multiplicative measure on $D(X)$ with parameter $\theta = 1$.

This definition corresponds to the representation of a general process $\xi \in D$ as a difference of positive processes: $\xi = \xi_1 - \xi_2$, where $\xi_1, \xi_2 \in D^+$. We will use for this measure the same notation \mathcal{L}_θ indicating the space D or D^+ if needed.

Theorem 4.1 immediately implies the following statement. Consider the space

$$L^0 = \left\{ a : X \rightarrow \mathbb{R} : \int_X |\log |a(x)|| d\nu(x) < \infty \right\}.$$

THEOREM 4.4. *The multiplicative measure \mathcal{L}_θ on the space $D(X, \nu)$ is projective invariant with respect to the group L^0 of multipliers, and the density is given by the formula*

$$\frac{dM_a(\mathcal{L}_\theta)}{d\mathcal{L}_\theta} = \exp \left(-\theta \int_X \log |a(x)| \, d\nu(x) \right). \quad (15)$$

COROLLARY 4.4. *Multiplicative measures on the space $D(X, \nu)$ are invariant with respect to the subgroup $L_0^0 = \{a \in L^0 : \int_X \log |a(x)| \, d\nu(x) = 0\}$.*

5. A MODEL OF THE CANONICAL REPRESENTATION OF THE CURRENT GROUP $SL(2, F)$

5.1. The Canonical Representation of the Current Group

Now we can return to the canonical representation of the current group $SL(2, F)$, where F is the space of measurable functions which was first described in [14], see also [15, 16, 5].

DEFINITION 5.1. The current group $SL(2, F)$ (or $SL(2, \mathbb{R})^X$) on the standard Borel space (X, ν) with fixed finite measure ν is the group of Borel bounded functions $T: X \rightarrow SL(2, \mathbb{R})$. Thus an element of this group is a 2×2 -matrix whose elements are bounded measurable scalar (real) functions on X .

Remark 5.1. The space (X, ν) is of course a Lebesgue space, but we prefer to consider individual functions (rather than classes mod 0).

The main idea of our construction is to extend to the case of non-commutative groups the construction of Lévy processes which starts from an infinitely divisible distribution and defines the Laplace transform of the corresponding Lévy process via the Laplace transform of this distribution (see Section 2). In a similar way, in order to define an (infinitely divisible) state on the current group G^X it suffices to define an infinitely divisible state on the group G .

In our case, the canonical representation of the current group $G^X = SL(2, \mathbb{R})^X$ is a unitary irreducible representation with spherical function $\Psi(g(\cdot)) = \langle U(g(\cdot)) \phi_0, \phi_0 \rangle$ (where ϕ_0 is a cyclic (vacuum) vector) given by the formula

$$\Psi(g(\cdot)) = C \exp \left(- \int_X \log (2 + \text{Tr}(g(x) g^*(x))) \, d\nu(x) \right), \quad g(\cdot) \in G^X.$$

The restriction of this spherical function onto the subgroup of constant functions (isomorphic to $G = \mathrm{SL}(2, \mathbb{R})$) equals $\Psi_0(g) = \frac{C}{2 + \mathrm{Tr} \, gg^*}$, the so-called *canonical state* of $\mathrm{SL}(2, \mathbb{R})$, see [14].

There are several constructions of the canonical irreducible representation of $\mathrm{SL}(2, F)$, see [5]. The most important of these constructions are the following ones.

1. The construction in the Fock space with cocycle in the space of the tail representation of $\mathrm{SL}(2, \mathbb{R})$, which is equivalent to the construction with gaussian measure.
2. The inductive limit of tensor products of representations of the complementary series.

A general construction of the representation of the current group G^X in the Fock space is described in [1]. This construction requires the existence of a non-trivial cocycle of the group G taking values in an irreducible unitary representation. For $\mathrm{SU}(n, 1)$ and $\mathrm{SO}(n, 1)$ (in particular for $\mathrm{SL}(2, \mathbb{R})$) such a cocycle was found in [15]. Note that the Fock space model is not commutative with respect to any natural subgroup of the group G^X , that is, no one natural subgroup is diagonalized in this model.

5.2. A Model of the Canonical Representation with Multiplicative Measures

Another realization of the canonical representation, namely a commutative model with respect to the unipotent subgroup, was given in [5]. This construction leads to the consideration of the L^2 -space over the law of the gamma process and then to the L^2 -space over the σ -finite measure which we now call multiplicative (in particular Lebesgue) measure in the space D .

Now we can go further and combine the inductive limit construction with a direct interpretation of the complementary series representations of $\mathrm{SL}(2, \mathbb{R})$ using multiplicative measures on the real line. Note that the restriction of the spherical function Ψ onto the unipotent subgroup $(\begin{smallmatrix} 1 & 0 \\ b(\cdot) & 1 \end{smallmatrix})$ of $\mathrm{SL}(2, \mathbb{R})^X$ equals

$$\Psi_{\mathrm{unip}}(b(\cdot)) = C \exp \left(- \int_X \log(4 + b^2(x)) \, d\nu(x) \right).$$

This is a positive definite function, thus it is the Fourier transform of some measure on X . It is easy to see that this measure is just a multiplicative measure on the space (X, ν) .

It is more convenient to describe first the representation of the triangular current group and then to extend it to the whole group $\mathrm{SL}(2, F)$. This approach is justified by the following fact: the restriction of the canonical representation of the current group $\mathrm{SL}(2, F)$ onto the triangular subgroup is still irreducible. This is similar to the well-known fact that all irreducible

representations of the principal and complementary series of $\mathrm{SL}(2, \mathbb{R})$ also give an irreducible restriction onto the ordinary triangular subgroup. In both finite-dimensional and infinite-dimensional cases the only element we need to define additionally is the element $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We leave this definition for another paper.

So consider first the triangular subgroup \mathcal{T} of $\mathrm{SL}(2, F)$,

$$\mathcal{T} = \left\{ T_{a,b} = \begin{pmatrix} a(\cdot)^{-1} & 0 \\ b(\cdot) & a(\cdot) \end{pmatrix} \right\}.$$

THEOREM 5.1. *The formula*

$$\mathcal{U}(T_{a,b}) F(\eta) = \exp \left(\int_X \log |a(x)| \, dv(x) + i \int_X a(x) b(x) \, d\eta(x) \right) F(M_a \eta) \quad (16)$$

defines a unitary irreducible representation of the triangular subgroup \mathcal{T} in the space $L^2(D, \mathcal{L}_1)$ which is extendable to a (unitary irreducible) representation of the whole group $\mathrm{SL}(2, F)$.

The main idea here is to use the same inductive limit construction starting from the tensor products of complementary series representations as in [5]. But instead of a Hilbert space with a complicated scalar product we use the tensor product of the L^2 -spaces $L^2(\mathbb{R}, \lambda_\theta)$ over one-dimensional multiplicative measures. This corresponds to the definition of the infinite-dimensional multiplicative measures by their finite-dimensional projections (14). Let us describe briefly this construction following [5].

For $\tau \in (0, 1)$, let T_τ be the complementary series representation of $G = \mathrm{SL}(2, \mathbb{R})$. This representation acts on the L^2 -space $L^2(\mathbb{R}, \lambda_\tau)$ over the multiplicative measure λ_τ by the following operators

$$\begin{aligned} \left(T_\tau \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \phi \right) (t) &= e^{i\tau b t} \phi(t), \\ \left(T_\tau \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \phi \right) (t) &= |a|^\tau \phi(a^2 t), \\ \left(T_\tau \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \phi \right) (t) &= \int_{-\infty}^{\infty} K_\tau(t, s) \phi(s) \, ds, \end{aligned}$$

where

$$K_\tau(t, s) = \frac{\tau}{2\pi} \int_{-\infty}^{\infty} |u|^{\tau-2} e^{i\tau(tu+su^{-1})} \, du.$$

Assume that the parameter measure ν is normalized, $|\nu| = 1$. For every finite measurable partition $\pi: X = X_1 \cup \dots \cup X_n$ of the base space X , where $\nu(X_k) = \tau_k$, let $H_\pi = H_{\tau_1} \otimes \dots \otimes H_{\tau_n}$, that is the L^2 -space over the product $\lambda_\pi = \lambda_{\tau_1} \dots \lambda_{\tau_n}$ of multiplicative measures. We identify this space with the space of functions $\phi: X \rightarrow \mathbb{R}$ which are constant on each X_k . Let $G_\pi = G_{X_1} \times \dots \times G_{X_n}$, where G_{X_k} is the subgroup of G isomorphic to $\text{SL}(2, \mathbb{R})$ and consisting of functions which are constant on X_k and equal to 1 outside X_k .

Note that if a partition π_2 is a refinement of the partition π_1 , then we have a natural embedding $G_{\pi_1} \hookrightarrow G_{\pi_2}$ and a natural embedding $H_{\pi_1} \hookrightarrow H_{\pi_2}$ commuting with the action of G_{π_1} . Let $\tilde{G}^X = \varinjlim G_\pi$ and $H = \varinjlim H_\pi$. It is easy to see that the group \tilde{G}^X acts on the pre-Hilbert space H , and the restriction of this action on the triangular subgroup is given by (16). This representation is extendable to a unitary representation of the whole group $G^X = \text{SL}(2, F)$ in the Hilbert space \tilde{H} which is the completion of H , and this is the representation from Theorem 5.1.

The question about an isomorphism between the models of the canonical representation which are commutative with respect to different subgroups is related to the question about special isomorphisms between the L^2 -spaces over the distributions of different Lévy processes. This question is studied very poorly (except a well-studied case corresponding to the isomorphism between the Poisson and the Gaussian processes). We would only like to mention that the vacuum vector of the Fock representation corresponds in our realization to the function $\phi_0(\eta) = e^{-\eta(X)}$.

6. ZERO-STABLE MEASURES

6.1. One-Dimensional Zero-Stable Measures

In this section we show how one should reformulate the ordinary definition of α -stable laws for $\alpha \in (0, 2]$ in order to extend it to the case of σ -finite measures and to define zero-stable measures.

The first observation is that the standard definition of stability is equivalent to the following one. Let F be a distribution on \mathbb{R} . Consider the distribution $F \times F$ on $\mathbb{R} \times \mathbb{R}$.

DEFINITION 6.1. The law F is *stable*, if there exists a norm on the space $(\mathbb{R} \times \mathbb{R})^*$ of linear functionals on $\mathbb{R} \times \mathbb{R}$ such that any two linear functionals f_1 and f_2 on $\mathbb{R} \times \mathbb{R}$ of equal norms have the same distribution with respect to $F \times F$.

Note that in case of linear functionals f_1 and f_2 the equality of distributions is equivalent to the existence of a $F \times F$ -preserving transformation

$L: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ such that $f_2 = f_1 \circ L$. Thus we obtain the following definition of a stable law which applies to σ -finite measures.

DEFINITION 6.2. The measure F on \mathbb{R} (finite on compact sets) is called *stable*, if there exists a norm $\|\cdot\|$ on $(\mathbb{R} \times \mathbb{R})^*$ such that for any two linear functionals f_1 and f_2 on $\mathbb{R} \times \mathbb{R}$ of equal norms there exists a $F \times F$ -preserving transformation $L: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ such that $f_2 = f_1 \circ L$.

A well-known theorem (see, for example, [2, Ch. VI, Sect. 1]) shows that in case of finite measures F satisfying some natural conditions the norm $\|\cdot\|$ must be the α -norm for some $\alpha \in (0, 2]$, that is $\|f\|_\alpha = (|a_1|^\alpha + |a_2|^\alpha)^{1/\alpha}$ for $f(x_1, x_2) = a_1 x_1 + a_2 x_2$. The corresponding measure is called α -stable.

Let $a_1 a_2 \neq 0$. If $\alpha \rightarrow 0$, then

$$\begin{aligned} 2^{-1/\alpha} \|f\|_\alpha &= \left(\frac{|a_1|^\alpha + |a_2|^\alpha}{2} \right)^{1/\alpha} = \left(\frac{1}{2} |a_1|^\alpha + \frac{1}{2} |a_2|^\alpha \right)^{1/\alpha} \\ &= \left(\frac{1}{2} + \alpha \log |a_1| + O(\alpha^2) + \frac{1}{2} + \alpha \log |a_2| + O(\alpha^2) \right)^{1/\alpha} \\ &= (1 + \alpha \log |a_1 a_2| + O(\alpha^2))^{1/\alpha} \rightarrow \log |a_1 a_2|, \end{aligned}$$

thus it is natural to consider the quasi-norm $\|f\|_0 = |a_1 a_2|$ as a limit of α -norms when α tends to 0. We obtain the following definition of a zero-stable measure on \mathbb{R} .

DEFINITION 6.3. A σ -finite measure F on \mathbb{R} (finite on compact sets) is called *zero-stable*, if for each two linear functionals f_1 and f_2 on $\mathbb{R} \times \mathbb{R}$ with $\|f_1\|_0 = \|f_2\|_0 < \infty$, there exists a transformation $L: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ preserving $F \times F$ such that $f_2 = f_1 \circ L$.

Remark 6.1. Note that in case $\alpha = 2$, corresponding to the normal law F , the $F \times F$ -preserving transformation L from Definition 6.2 is a rotation. In the “opposite” case $\alpha = 0$ the corresponding transformation is hyperbolic, of the form $(a_1, a_2) \mapsto (ca_1, \frac{a_2}{c})$. In both cases it is a linear mapping. Unlike these two extreme cases, the symmetries for intermediate cases $\alpha \in (0, 2)$ are not known, but they cannot be linear.

PROPOSITION 6.1. For any $\beta < 1$, the measure on \mathbb{R} with density $|x|^{-\beta} dx$ is zero-stable. In particular, the Lebesgue measure on \mathbb{R} is zero-stable.

Proof. Easy calculation. ■

6.2. Zero-Stable Processes

Now we can give an analogue of the theory of σ -finite stable processes on arbitrary spaces.

Recall that each bounded Borel function a on X defines a linear functional f_a on the space D of finite discrete measures on X by $f_a(\eta) = \int_X a(x) d\eta(x)$. Let $\|\cdot\|_\alpha$ denote the α -norm $\|a\|_\alpha = (\int |a(x)|^\alpha d\nu(x))^{1/\alpha}$ for $\alpha > 0$, and let $\|a\|_0 = \exp(\int_X \log |a(x)| d\nu(x))$.

DEFINITION 6.4. The measure P_α on D is α -stable if for each two linear functionals f_{a_1} and f_{a_2} with $\|a_1\|_\alpha = \|a_2\|_\alpha < \infty$, there exists a P_α -preserving transformation $L : D \rightarrow D$ such that $f_{a_2} = f_{a_1} \circ L$.

For $\alpha > 0$ we obtain the ordinary theory of α -stable processes (see Section 6.3). But it is easy to check that multiplicative measures \mathcal{L}_θ satisfy this condition for $\alpha = 0$. Indeed, the condition $\|a_1\|_0 = \|a_2\|_0$ is equivalent to $\int_X \log(a_2/a_1)(x) d\nu(x) = 0$. Hence the multiplier M_{a_2/a_1} preserves the measure \mathcal{L}_θ by Corollary 4.1, and it is obvious that $f_{a_2} = f_{a_1} \circ M_{a_2/a_1}$. In fact, it is not difficult to deduce from Theorem 4.3 the following statement. Recall that a σ -finite measure on the space D^+ is called admissible if it is finite on compact sets and equivalent to the law of some Lévy process.

PROPOSITION 6.2. Let μ be an admissible measure on D^+ . Then μ is zero-stable if and only if it is multiplicative.

6.3. Lebesgue Measure as a Limit of Renormalized α -Stable Measures when α Tends to Zero

In this section we present another aspect of the fact that the infinite-dimensional Lebesgue measure is a natural limit of α -stable laws as α goes to zero. The first suggestion of this kind was formulated in [9], and the weak convergence result was formulated in [17].

Let $\alpha \in (0, 1)$. The standard α -stable process on the space (X, ν) is a Lévy process with Lévy measure

$$dA_\alpha = \frac{c\alpha}{\Gamma(1-\alpha)} s^{-\alpha-1} ds, \quad s > 0, \quad (17)$$

where $c > 0$ is an arbitrary fixed positive number.

The corresponding infinitely divisible law is the α -stable law F_α on \mathbb{R}_+ . The Laplace transform of the law P_α of the α -stable process equals

$$\mathbb{E}_\alpha \left[\exp \left(- \int_X a(x) d\eta(x) \right) \right] = \exp \left(- c \int_X a(x)^\alpha d\nu(x) \right), \quad (18)$$

for an arbitrary measurable function $a : X \rightarrow \mathbb{R}_+$ with $\int_X a(x)^\alpha d\nu(x) < \infty$.

Now consider the measure on D^+ which has a constant density $e^{1/\alpha}$ with respect to the α -stable law, and let \check{P}_α be the image of this measure under the multiplier $M_{\alpha^{-1/\alpha}}$ by a constant function $\alpha^{-1/\alpha}$, that is

$$\check{P}_\alpha = M_{\alpha^{-1/\alpha}}(e^{1/\alpha} P_\alpha).$$

In the following theorem by weak convergence we mean convergence on the class \mathcal{M} of test functions on which the Laplace transform of \mathcal{L}_1 is defined.

THEOREM 6.1. *The measures \check{P}_α converge weakly to the infinite-dimensional Lebesgue measure \mathcal{L}_1 .*

Proof. It is easy to show that the Laplace transform of the measure \hat{P}_α equals

$$\int_{D^+} \exp(-f_a(\eta)) d\check{P}_\alpha(\eta) = \exp\left(-\frac{1}{\alpha} \int_X ((a(x))^\alpha - 1) dv(x)\right).$$

But $\frac{1}{\alpha}((a(x))^\alpha - 1) = \frac{1}{\alpha}(\alpha \log a(x) + o(\alpha)) \rightarrow \log a(x)$ as $\alpha \rightarrow 0$, hence $\int_{D^+} \exp(-f_a(\eta)) d\check{P}_\alpha(\eta)$ tends to

$$\exp\left(-\int_X \log a(x) dv(x)\right) = \int_{D^+} \exp(-f_a(\eta)) d\mathcal{L}_1(\eta),$$

and Theorem 7 follows. ■

A similar weak convergence result holds for the gamma measure too. Namely, consider the (probability) measure \tilde{P}_λ on D^+ equivalent to the α -stable measure P_α with density

$$\frac{d\tilde{P}_\alpha}{dP_\alpha}(\eta) = \frac{\exp(-\alpha^{-1/\alpha} \cdot \eta(X))}{\mathbb{E}_\alpha[\exp(-\alpha^{-1/\alpha} \cdot \eta(X))]} = e^{1/\alpha} \cdot e^{-\alpha^{-1/\alpha} \cdot \eta(X)}.$$

Let $\hat{P}_\alpha = M_{\alpha^{-1/\alpha}} \tilde{P}_\alpha$.

PROPOSITION 6.3. *The measures \hat{P}_α converge weakly to the gamma measure \mathcal{G}_1 when $\alpha \rightarrow 0$.*

APPENDIX: DECOMPOSITION THEOREM FOR LÉVY PROCESSES AND MEASURES OF PRODUCT TYPE ON THE CONE

Let C be the cone

$$C = \left\{ z = (z_1, z_2, \dots) : z_1 \geq z_2 \geq \dots \geq 0, \sum z_i < \infty \right\} \subset l^1.$$

We define a map $T: D^+ \rightarrow C \times X^\infty$ by

$$T\eta = ((Q_1, Q_2, \dots), (X_1, X_2, \dots)), \quad (\text{A.1})$$

if $\eta = \sum Q_i \delta_{X_i}$, where $Q_1 \geq Q_2 \geq \dots$.

DEFINITION A.1. Let P be a distribution on the space D^+ , and let η be a random process obeying the law P . The random sequence of charges Q_1, Q_2, \dots is called the *conic part* of the process η , and its distribution on the cone C is called the *conic part* of the law P .

Note that in view of representation (6) the conic part of the Lévy process with Lévy measure Λ is just the ordered sequence of points of the Poisson process on \mathbb{R}_+ with mean measure $|v| \Lambda$. Thus the conic part depends only on Λ and on the full charge of the parameter measure v . In fact, the following theorem shows that studying the Lévy process may be essentially reduced to studying its conic part, since the construction of the process includes the parameter measure in a trivial way. This fundamental property of Lévy processes is a particular case of the representation theorem first proved in [4]. A simpler proof of this fact is presented in [13].

THEOREM A.1. Let $\eta = \sum Q_i \delta_{X_i}$ be a homogeneous Lévy process on the space (X, v) with Lévy measure Λ . Let $\bar{v} = v/|v|$ be the normalized measure. Then $TP_\Lambda = \kappa_{|v|\Lambda} \times \bar{v}^\infty$, i.e. X_1, X_2, \dots is a sequence of i.i.d. random variables with common distribution v , and this sequence is independent of the conic part $\{Q_i\}_{i \in \mathbb{N}}$.

We now define a special class of measures on C indexed by infinitely divisible distributions on the half-line. Fix an integer $n \in \mathbb{N}$ and a probability vector $p = (p_1, \dots, p_n)$ (i.e. a vector p with $p_1, \dots, p_n > 0$ and $p_1 + \dots + p_n = 1$). Consider a sequence ξ_i of i.i.d. variables such that $P(\xi_i = k) = p_k$ for $k = 1, \dots, n$. For $Q = (Q_1, Q_2, \dots) \in C$, denote by $\Sigma_k^{(p)} = \Sigma_k^{(p)}(Q)$ the random sum $\Sigma_k^{(p)} = \sum_{i: \xi_i = k} Q_i$. Let Q be a random series with distribution κ on C such that the distribution F of the sum $\sum Q_i$ is infinitely divisible.

DEFINITION A.2. We say that a series Q (and its distribution κ) is of *product type*, if for each $n \in \mathbb{N}$ and each probability vector p the sums $\Sigma_1^{(p)}, \dots, \Sigma_n^{(p)}$ are independent and $\Sigma_k^{(p)}$ obeys the law F^{*p_k} .

THEOREM A.2. The measure κ on the cone C is the conic part of some Lévy process P_Λ with Lévy measure Λ satisfying (1)–(4) if and only if it is of product type with $F = F_\Lambda$.

A more detailed exposition of this section can be found in [13].

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